LIE ALGEBRAS WITH GENERALIZED ASSOCIATIVE STRUCTURES

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Abstract. We investigate Lie algebras whose Lie bracket is also associative or cubic associative. This allows the characterization of a class of nilpotent Lie algebras with a given nilindex. We generalize the notion of associativity considering f-associativity, notion closed to the notion of Hom-(f)-associativity.

2000 MSC: 17B25 , 16W10 , 18D50

1. Associative Lie multiplication

Let $\mathfrak{g}$ be a Lie algebra over a field of characteristic 0. If we denote by $[X,Y]$ the Lie bracket of $\mathfrak{g}$, it satisfies the following identities

$$\begin{cases} [X,Y] = -[Y,X], \\ [[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0 \quad \text{(Jacobi Identity)}, \end{cases}$$

for any $X,Y,Z \in \mathfrak{g}$. We assume moreover that the Lie bracket is also an associative product, that is, it satisfies

$$[[X,Y],Z] = [X,[Y,Z]],$$

for any $X,Y,Z \in \mathfrak{g}$. The Jacobi Identity therefore implies

$$[[Z,X]],Y] = 0.$$

Proposition 1. *The Lie bracket of the Lie algebra $\mathfrak{g}$ is an associative product if and only if $\mathfrak{g}$ is a two-step nilpotent Lie algebra.*

In fact, the relation $[[Z,X],Y] = 0$ means that $[[\mathfrak{g},\mathfrak{g}],\mathfrak{g}] = \mathcal{C}^2(\mathfrak{g}) = 0$ where $\mathcal{C}^i(\mathfrak{g})$ denotes the ideals of the descending central sequence of $\mathfrak{g}$. The converse is obvious.

We know the classification of complex two-step nilpotent Lie algebras up to the dimension 7. In the following our notation and terminology will be based on [5]

Any two-step nilpotent complex non abelian and indecomposable Lie algebra of dimension less than 7 is isomorphic to one of the following algebras:

(1) For the dimensions less than or equal to 3:
   - $\mathfrak{n}_3^1 = \mathfrak{h}_3 \ : \ [X_1,X_2] = X_3.$

(2) In dimension 5:
   - $\mathfrak{n}_5^5 : \ [X_1,X_2] = X_3, \ [X_1,X_4] = X_5;$
   - $\mathfrak{n}_5^6 =$ the Heisenberg algebra $\mathfrak{h}_2 : \ [X_1,X_2] = X_3, \ [X_4,X_5] = X_3.$

(3) In dimension 6:
   - $\mathfrak{n}_6^{19} : \ [X_1,X_i] = X_{i+1}, \ i = 2,4, \ [X_2,X_6] = X_5;$
   - $\mathfrak{n}_6^{20} : \ [X_1,X_i] = X_{i+1}, \ i = 2,4, \ [X_2,X_4] = X_6.$

(4) In dimension 7:

- $\mathfrak{n}_7^{120}$: $[X_1, X_i] = X_{i+1},\ i = 2,4,6,\quad [X_2, X_4] = X_7;$
- $\mathfrak{n}_7^{121}$: $[X_1, X_i] = X_{i+1},\ i = 2,4,6;$
- $\mathfrak{n}_7^{122}$: $[X_1, X_i] = X_{i+1},\ i = 2,4,6,\quad [X_4, X_6] = X_7;$
- $\mathfrak{n}_7^{123}$: $[X_1, X_i] = X_{i+1},\ i = 2,4,6,\quad [X_2, X_4] = X_5,\ [X_4, X_6] = X_3;$
- $\mathfrak{n}_7^{124}$: $[X_1, X_i] = X_{i+1},\ i = 2,4,\quad [X_6, X_7] = X_5,\ [X_4, X_7] = X_3;$
- $\mathfrak{n}_7^{134}$: $[X_1, X_i] = X_{i+1},\ i = 2,4,\quad [X_6, X_7] = X_5;$
- $\mathfrak{n}_7^{126}$: $[X_1, X_2] = X_3,\ [X_4, X_5] = X_3,\ [X_6, X_7] = X_3.$

To develop the operadic point of view, let us recall that an operad is a sequence $\mathcal{P} = \{\mathcal{P}(n), n \in \mathbb{N}^*\}$ of $\mathbb{K}[\Sigma_n]$-modules, where $\mathbb{K}[\Sigma_n]$ is the algebra group associated with the symmetric group Σ_n, with $comp_i$-operations (see [12]). The main example corresponds to the free operad $\Gamma(E) = \{\Gamma(E)(n)\}$ generated by a $\mathbb{K}[\Sigma_2]$-module. An operad $\mathcal{P}$ is called binary quadratic if there is a $\mathbb{K}[\Sigma_2]$-module E and a $\mathbb{K}[\Sigma_3]$-submodule R of $\Gamma(E)(3)$ such that $\mathcal{P}$ is isomorphic to $\Gamma(E)/\mathcal{R}$ where $\mathcal{R}$ is the operadic ideal generated by $\mathcal{R}(3) = R$.

Proposition 2. *There exists a binary quadratic operad, denoted by $2\mathcal{N}ilp$, with the property that any $2\mathcal{N}ilp$-algebra is a 2-step nilpotent Lie algebra.*

In fact, we consider $E = sgn_2$, that is, the representation of Σ_2 by the signature, then $\Gamma(E)(3) = sgn_3 \oplus V_2$ where $V_2 = \{(x, y, z) \in \mathbb{K}^3, x + y + z = 0\}$. Let R be the submodule of $\Gamma(E)(3)$ generated by $(x_i \cdot x_j) \cdot x_k$, i, j, k all different. We deduce that $2\mathcal{N}ilp(2)$ is the $\mathbb{K}[\Sigma_2]$-module generated by $x_1 \cdot x_2$ with the relation $x_2 \cdot x_1 = -x_1 \cdot x_2$ and it is a 1-dimensional vector space and $2\mathcal{N}ilp(3) = \{0\}$.

Proposition 3. *The operad $2\mathcal{N}ilp$ is Koszul.*

Recall some general definitions and results on the duality of a binary quadratic operad. The generating function of a binary quadratic operad $\mathcal{P}$ is

$$g_{\mathcal{P}}(x) = \sum_{a \geq 1} \frac{1}{a!} \dim(\mathcal{P}(a)) x^a.$$

Thus the generating function of the operad $2\mathcal{N}ilp$ is the polynomial

$$g_{2\mathcal{N}ilp}(x) = x + \frac{x^2}{2}.$$

The dual operad $\mathcal{P}^!$ of the operad $\mathcal{P}$ is the quadratic operad $\mathcal{P}^! := \Gamma(E^\vee)/(R^\perp)$, where $R^\perp \subset \Gamma(E^\vee)(3)$ is the annihilator of $R \subset \Gamma(E)(3)$ in the pairing

$$(1)\qquad \left\{ \begin{array}{l} < (x_i \cdot x_j) \cdot x_k, (x_{i'} \cdot x_{j'}) \cdot x_{k'} >= 0, \text{ if } \{i, j, k\} \neq \{i', j', k'\}, \\ < (x_i \cdot x_j) \cdot x_k, (x_i \cdot x_j) \cdot x_k >= (-1)^{\varepsilon(\sigma)}, \\ \qquad\qquad \text{with } \sigma = \left(\begin{array}{ccc} i & j & k \\ i' & j' & k' \end{array} \right) \text{ if } \{i, j, k\} = \{i', j', k'\}, \\ < x_i \cdot (x_j \cdot x_k), x_{i'} \cdot (x_{j'} \cdot x_{k'}) >= 0, \text{ if } \{i, j, k\} \neq \{i', j', k'\}, \\ < x_i \cdot (x_j \cdot x_k), x_i \cdot (x_j \cdot x_k) >= -(-1)^{\varepsilon(\sigma)}, \\ \qquad\qquad \text{with } \sigma = \left(\begin{array}{ccc} i & j & k \\ i' & j' & k' \end{array} \right) \text{ if } (i, j, k) = (i', j', k'), \\ < (x_i \cdot x_j) \cdot x_k, x_{i'} \cdot (x_{j'} \cdot x_{k'}) >= 0, \end{array} \right.$$

and $(R^{\perp})$ is the operadic ideal generated by $R^{\perp}$. We deduce

$$\dim(2\mathcal{N}ilp)^!(1) = 1,\ \dim(2\mathcal{N}ilp)^!(2) = 1,\ \dim(2\mathcal{N}ilp)^!(3) = 3,\ \dim(2\mathcal{N}ilp)^!(4) = 15$$

and more generally, if we denote by d_k the dimension of $(2\mathcal{N}ilp)^!(k)$, we have

$$\begin{cases} d_{2k+1} = \displaystyle\sum_{i=1}^{k} C_{2k+1}^{i} d_i d_{2k+1-i}, \\ d_{2k} = \displaystyle\sum_{i=1}^{k-1} C_{2k}^{i} d_i d_{2k-i} + \frac{1}{2} C_{2k}^{k} d_k^2. \end{cases}$$

In fact, the dual operad is $\Gamma(\mathbb{1})$ the free operad generated by a commutative operation. The generating function of $2\mathcal{N}ilp^!$ is

$$\sum_{k\geq 1} \frac{d_k}{k!} x^k.$$

and the generating function $g_{2\mathcal{N}ilp}$ of the operad $2\mathcal{N}ilp$ satisfies the functional equation

$$g_{2\mathcal{N}ilp}(-g_{2\mathcal{N}ilp^!}(-x)) = x.$$

If an operad $\mathcal{P}$ is Koszul, then its dual $\mathcal{P}^!$ is also Koszul and the generating functions are related by the functional equation

$$g_{\mathcal{P}}(-g_{\mathcal{P}^!}(-x)) = x.$$

It is known that $\Gamma(\mathbb{1})$ is Koszul, so also $2\mathcal{N}ilp$ and this implies the proposition.

Remarks.

(1) Recall that the operad satisfies the Koszul property if the corresponding free algebra is Koszul, that is, its natural or operadic homology is trivial except in degree 0. If $\mathcal{L}_r$ is the free Lie algebra of rank r (i.e. on k (free) generators), and if $\mathcal{C}^3(\mathcal{L}_r)$ is the third part of its descending central series, thus the free two-step nilpotent Lie algebra of rank r is $\mathcal{N}(2,r) = \mathcal{L}_r/\mathcal{C}^3(\mathcal{L}_r)$. If V_r is the r-dimensional vector space corresponding to the homogeneous component of degree 1 of $\mathcal{L}_r$, thus $\mathcal{N}(2,r) = V_r \oplus \bigwedge^2(V_r)$. For example, if $r = 2$, thus $\mathcal{N}(2,2)$ is a 3-dimensional Lie algebra with basis $\{e_1, e_2, e_1 \wedge e_2\}$, where the last vector corresponds to $[e_1, e_2]$. If $r = 3$, thus $\mathcal{N}(2,3)$ is the 6-dimensional Lie algebra generated by $\{e_1, e_2, e_3, e_1\wedge e_2, e_1\wedge e_3, e_2\wedge e_3\}$. We can define an homology of $\mathcal{N}(2,r)$ using the standard complex $(\bigwedge^*(\mathcal{N}(2,r)), \partial_*)$ where

$$\partial_p : \bigwedge^{p}(\mathcal{N}(2,r)) \to \bigwedge^{p-1}(\mathcal{N}(2,r))$$

is defined by

$$\partial_p(x_1 \wedge x_2 \wedge \cdots \wedge x_p) = \sum_{i<j} (-1)^{i+j+1} [x_i, x_j] \wedge x_1 \wedge \cdots \widehat{x_i} \cdots \widehat{x_j} \cdots \wedge x_p$$

and $\partial_p = 0$ is $p \leq 0$. Let m_p be the dimension of the pth homology space $Ker\,\partial_p/Im\,\partial_{p+1}$. For example if $r = 2$, $\dim \mathcal{N}(2,2) = 3$ and $\mathcal{N}(2,2)$ is generated by $e_1, e_2, [e_1, e_2] = e_3$. It is isomorphic to the Heisenberg algebra. We have $m_0 = 1, m_1 = m_2 = 2, m_3 = 1$. The general case was studied in [19]. The homology spaces are never trivial. For this complex, the free 2-step nilpotent Lie algebra is not Koszul.

(2) As a nilpotent Lie algebra is unimodular, we have the Poincaré duality. This implies that the second cohomology space of the free 2-step nilpotent algebra of rank r is trivial and this algebra is rigid in the variety of 2-step nilpotent Lie algebra of dimension $r(r-1)/2$. It defines an open orbit and an algebraic component in this variety.

Remarks

(1) Let us consider an associative algebra $(A, \cdot)$ where $x \cdot y$ denotes the multiplication in A. Thus
$$[x, y] = x \cdot y - y \cdot x$$
is a Lie bracket. This Lie bracket is associative if and only if the multiplication of A satisfies
$$(x \cdot y) \cdot z - (y \cdot x) \cdot z - (z \cdot x) \cdot y + (z \cdot y) \cdot x = 0.$$
Let v be the vector of $\mathbb{K}[\Sigma_3]$ $v = Id - \tau_{12} + \tau_{13} - c^2$ where τ_{ij} is the transposition which exchanges the two elements i and j and c the 3-cycle (123). The orbit of v with respect to the action of the group Σ_3 generates a 2-dimensional vector space with basis $\{v, \tau_{13} \cdot v\}$. We deduce that these algebras can be considered as $\mathcal{P}$-algebras where the quadratic operad $\mathcal{P}$ is defined by $\mathcal{P}(2) = sgn_2$ and $\mathcal{P}(3) = \Gamma(E)(3)/R$ with R the submodule generated by $v((x_1x_2)x_3)$ and $\tau_{13} \cdot v((x_1x_2)x_3)$ with $\tau((x_1x_2)x_3) = ((x_{\tau(1)}x_{\tau(2)})x_{\tau(3)})$ for any $\tau \in \Sigma_3$. In particular $\dim \mathcal{P}(3) = 4$.

(2) A Pre-Lie algebra is a non associative algebra defined by the identity
$$(xy)z - x(yz) = (xz)y - x(zy)$$
for all x, y, z. Assume that the Lie bracket of $\mathfrak{g}$ satisfies also the Pre-Lie identity, that is,
$$[[x, y], z] - [x, [y, z]] = [[x, z], y] - [x, [z, y]].$$
Applying anticommutativity to this equation we obtain
$$[[x, y], z] + [[y, z], x] + [[z, x], y] + [[y, z], x] = 0;$$
and finally the Jacobi identity gives
$$[[y, z], x] = 0.$$
This shows that the Lie algebra is also 2-step nilpotent and the Lie bracket is an associative product.

(3) In [6], we have defined classes of non associative algebras including in particular Pre-Lie algebras, Lie-admissible algebras and more generally algebras with a non associative defining identity admitting a symmetry with respect to a subgroup of the symmetric group Σ_3. These algebras have been called G_i-associative algebras where G_i, $i = 1, \cdots, 6$ are the subgroups of Σ_3. More precisely, a $G_1 = \{Id\}$-associative algebra is an associative algebra, a $G_2 = \{Id, \tau_{12}\}$-associative algebra is a Vinberg algebra that is, its multiplication satisfies
$$(xy)z - x(yz) = (yx)z - y(xz),$$
a $G_3 = \{Id, \tau_{23}\}$-associative algebra is a Pre-Lie algebra, a $G_4 = \{Id, \tau_{13}\}$-associative algebra satisfies
$$(xy)z - x(yz) = (zy)x - z(yx),$$

a $G_5 = \{Id, c, c^2\}$-associative algebra satisfies

$$(xy)z - x(yz) + (yz)x - y(zx) + z(xy) - z(xy) = 0,$$

and a $G_6 = \Sigma_3$-associative algebra is a Lie-admissible algebra. While writing this paper we discover that this notion already appear in [17]. It is easy to see that if the Lie bracket of $\mathfrak{g}$ satisfies the G_i-associativity for $i = 1, 2, 3$, or 4 then $\mathfrak{g}$ is 2-step nilpotent and the Lie bracket is an associative multiplication. The defining equations associated to the cases $i = 5$ and 6 are always satisfied because the G_5-conditions corresponds to the Jacobi identity and a Lie algebra is, in particular, a Lie-admissible algebra.

2. Cubic associative Lie multiplication

Let A be a $\mathbb{K}$ associative algebra with binary multiplication xy. The associativity which is the quadratic relation

$$(xy)z = x(yz)$$

implies six cubic relations

$$\begin{cases} ((xy)z)t = (x(yz))t, \\ (x(yz))t = x((yz)t), \\ x((yz)t) = x(y(zt)), \\ x(y(zt)) = (xy)(zt), \\ (xy)(zt) = ((xy)z)t. \end{cases} \quad (*)$$

Recall that these relations correspond to the edges of the Stasheff pentagon.

Definition 4. *A binary algebra, that is, an algebra whose multiplication is given by a bilinear map, is called cubic associative if the multiplication satisfies the cubic relations* $(*)$.

We call these relations cubic because if we denote by μ the multiplication, it occurs exactly three times in each term of the relations. For example, the first relation writes as

$$\mu \circ (\mu \circ (\mu \otimes Id) \otimes Id) = \mu \circ (\mu \circ (Id \otimes \mu) \otimes Id)$$

which is cubic in μ. It is the same thing for all other relations.

If $\mathcal{A}ss = \Gamma(E)/(R_{\mathcal{A}ss})$ is the operad for associative algebras, the relations $(*)$ are the generating relations of $(R_{\mathcal{A}ss})(4)$. But these relations are, following from the relations defining $(R_{\mathcal{A}ss})(3) = R_{\mathcal{A}ss}$. In Definition 4, we do not assume that the algebra is associative. It is clear that $(*)$ do not implies associativity. From the relations $(*)$ we can define a binary cubic operad $\mathcal{A}ss\mathcal{C}ubic$. This operad will be studied in the next paragraph.

Proposition 5. *Let $\mathfrak{g}$ be a Lie algebra. The Lie bracket is cubic associative if and only if $\mathfrak{g}$ is 3-step nilpotent.*

In fact, the first identity of $(*)$ becomes

$$\begin{array}{rcl} [[[X_1, X_2], X_3], X_4] & = & [[X_1, [X_2, X_3]], X_4] \\ & = & -[[[X_2, X_3]], X_1], X_4] \end{array}$$

and finally

$$[[[X_1, X_2], X_3], X_4] + [[[X_2, X_3]], X_1], X_4] = [[[X_3, X_1]], X_2], X_4] = 0,$$

which implies that $\mathfrak{g}$ is 3-nilpotent. Conversely, if $\mathfrak{g}$ is 3-nilpotent, all the relations of $(*)$ are satisfied.

The classification of the 3-step nilpotent Lie algebra of dimension less than 7 is the following

Dimension 4

$\mathfrak{n}_4^1$: $[X_1, X_i] = X_{i+1},\ i = 2, 3.$

Dimension 5

$\mathfrak{n}_5^3$: $[X_1, X_i] = X_{i+1},\ i = 2, 3,\ [X_2, X_5] = X_4;$

$\mathfrak{n}_5^4$: $[X_1, X_i] = X_{i+1},\ i = 2, 3,\ [X_2, X_3] = X_5.$

Dimension 6

$\mathfrak{n}_6^{11}$: $[X_1, X_i] = X_{i+1},\ i = 2, 3, 5,\ [X_5, X_6] = X_4;$

$\mathfrak{n}_6^{12}$: $[X_1, X_i] = X_{i+1},\ i = 2, 3, 5,\ [X_2, X_5] = X_4;$

$\mathfrak{n}_6^{13}$: $[X_1, X_i] = X_{i+1},\ i = 2, 3, 5,\ [X_2, X_3] = X_6,\ [X_2, X_5] = X_6;$

$\mathfrak{n}_6^{14}$: $[X_1, X_i] = X_{i+1},\ i = 2, 3, 5,\ [X_2, X_3] = X_4 - X_6,\ [X_2, X_5] = X_6;$

$\mathfrak{n}_6^{15}$: $[X_1, X_i] = X_{i+1},\ i = 2, 3, 5,\ [X_2, X_5] = X_6,\ [X_5, X_6] = X_4;$

$\mathfrak{n}_6^{16}$: $[X_1, X_i] = X_{i+1},\ i = 2, 3, 5,\ [X_2, X_3] = X_4;$

$\mathfrak{n}_6^{17}$: $[X_1, X_i] = X_{i+1},\ i = 2, 3, 5;$

$\mathfrak{n}_6^{18}$: $[X_1, X_i] = X_{i+1},\ i = 2, 3,\ [X_5, X_6] = X_4;$

Dimension 7

$\mathfrak{n}_7^{77}:\ [X_1, X_i] = X_{i+1},\ i = 2, 3, 5, 6,\quad [X_2, X_5] = X_7;$

$\mathfrak{n}_7^{78}:\ [X_1, X_i] = X_{i+1},\ i = 2, 3, 5, 6,\quad [X_2, X_6] = X_4,\ [X_2, X_5] = X_3;$

$\mathfrak{n}_7^{79}:\ [X_1, X_i] = X_{i+1},\ i = 2, 3, 5, 6,\quad [X_5, X_6] = X_4,\ [X_2, X_5] = X_7;$

$\mathfrak{n}_7^{80}:\ [X_1, X_i] = X_{i+1},\ i = 2, 3, 5, 6;$

$\mathfrak{n}_7^{81}:\ [X_1, X_i] = X_{i+1},\ i = 2, 3, 5, 6,\quad [X_5, X_6] = X_4;$

$\mathfrak{n}_7^{82}:\ [X_1, X_i] = X_{i+1},\ i = 2, 3, 5, 6,\quad [X_5, X_6] = X_4,\ [X_2, X_3] = X_7;$

$\mathfrak{n}_7^{83}:\ [X_1, X_i] = X_{i+1},\ i = 2, 3, 5, 6,\quad [X_5, X_6] = X_7,\ [X_2, X_3] = X_4 + X_7;$

$\mathfrak{n}_7^{84}:\ [X_1, X_i] = X_{i+1},\ i = 2, 3, 5, 6,\quad [X_5, X_6] = X_7,\ [X_2, X_3] = X_4;$

$$\mathfrak{n}_7^{85}: \begin{cases} [X_1, X_i] = X_{i+1},\ i = 2, 3, 5, 6,\ [X_3, X_5] = X_7,\ [X_2, X_5] = X_4 + X_6, \\ [X_2, X_3] = X_4; \end{cases}$$

$\mathfrak{n}_7^{86}:\ [X_1, X_i] = X_{i+1},\ i = 2, 3, 5, 6,\quad [X_5, X_6] = X_7,\ [X_2, X_3] = X_7;$

$$\mathfrak{n}_7^{87}: \begin{cases} [X_1, X_i] = X_{i+1},\ i = 2, 3, 5, 6,\quad [X_5, X_6] = X_7 + X_4,\ [X_2, X_6] = X_4, \\ [X_2, X_5] = X_3; \end{cases}$$

$$\mathfrak{n}_7^{88}: \begin{cases} [X_1, X_i] = X_{i+1},\ i = 2, 3, 5, 6,\quad [X_5, X_6] = X_4,\ [X_3, X_5] = X_7, \\ [X_2, X_3] = X_4,\ [X_2, X_5] = X_6; \end{cases}$$

$\mathfrak{n}_7^{89}:\begin{cases}[X_1,X_i]=X_{i+1},\ i=2,3,5,6,\quad [X_5,X_6]=X_4,\ [X_2,X_3]=X_4,\\ [X_2,X_5]=X_7;\end{cases}$

$\mathfrak{n}_7^{90}:\begin{cases}[X_1,X_i]=X_{i+1},\ i=2,3,5,6,\quad [X_2,X_3]=X_4,\ [X_3,X_5]=X_7,\\ [X_2,X_5]=X_6;\end{cases}$

$\mathfrak{n}_7^{91}:\ [X_1,X_i]=X_{i+1},\ i=2,3,5,6,\quad [X_5,X_6]=X_7;$

$\mathfrak{n}_7^{92}:\ [X_1,X_i]=X_{i+1},\ i=2,3,5,6,\quad [X_2,X_3]=X_4,\ [X_2,X_5]=X_7;$

$\mathfrak{n}_7^{93}:\ [X_1,X_i]=X_{i+1},\ i=2,3,5,\quad [X_2,X_5]=X_7;$

$\mathfrak{n}_7^{94}:\ [X_1,X_i]=X_{i+1},\ i=2,3,5,\quad [X_2,X_5]=X_4,\ [X_2,X_3]=X_7;$

$\mathfrak{n}_7^{95}:\ [X_1,X_i]=X_{i+1},\ i=2,3,5\ [X_2,X_3]=X_7;$

$\mathfrak{n}_7^{96}:\ [X_1,X_i]=X_{i+1},\ i=2,3,5,\quad [X_2,X_5]=X_7,\ [X_2,X_6]=X_4;$

$\mathfrak{n}_7^{97}:\ [X_1,X_i]=X_{i+1},\ i=2,3,5,\quad [X_2,X_6]=X_4,\ [X_3,X_5]=-X_4,\ [X_2,X_5]=X_7;$

$\mathfrak{n}_7^{98}:\begin{cases}[X_1,X_i]=X_{i+1},\ i=2,3,5,\quad [X_2,X_6]=X_4,\ [X_3,X_5]=-X_4,\\ [X_2,X_5]=X_7,\ [X_5,X_6]=X_4;\end{cases}$

$\mathfrak{n}_7^{99}:\ [X_1,X_i]=X_{i+1},\ i=2,3,5,\quad [X_2,X_7]=X_6,\ [X_2,X_3]=X_4;$

$\mathfrak{n}_7^{100}:\ [X_1,X_i]=X_{i+1},\ i=2,3,5,\quad [X_2,X_7]=X_4,\ [X_5,X_7]=X_6;$

$\mathfrak{n}_7^{101}:\ [X_1,X_i]=X_{i+1},\ i=2,3,5,\quad [X_2,X_7]=X_6,\ [X_5,X_7]=X_4;$

$\mathfrak{n}_7^{102}:\ [X_1,X_i]=X_{i+1},\ i=2,3,5,\quad [X_5,X_7]=X_4;$

$\mathfrak{n}_7^{103}:\ [X_1,X_i]=X_{i+1},\ i=2,3,5,\quad [X_2,X_7]=X_4;$

$\mathfrak{n}_7^{104}:\ [X_1,X_i]=X_{i+1},\ i=2,3,5,\quad [X_5,X_7]=X_4,\ [X_2,X_3]=X_4;$

$\mathfrak{n}_7^{105}:\ [X_1,X_i]=X_{i+1},\ i=2,3,5,\quad [X_2,X_7]=X_4,\ [X_2,X_3]=X_4;$

$\mathfrak{n}_7^{106}:\ [X_1,X_i]=X_{i+1},\ i=2,3,5,\quad [X_2,X_7]=X_4,\ [X_5,X_6]=X_4;$

$\mathfrak{n}_7^{107}:\ [X_1,X_i]=X_{i+1},\ i=2,3,5,\quad [X_5,X_7]=X_3,\ [X_6,X_7]=X_4;$

$\mathfrak{n}_7^{108}:\ [X_1,X_i]=X_{i+1},\ i=2,3,5,\quad [X_2,X_7]=X_6,\ [X_2,X_3]=X_6;$

$\mathfrak{n}_7^{109}:\ [X_1,X_i]=X_{i+1},\ i=2,3,5,\quad [X_5,X_7]=X_6,\ [X_2,X_3]=X_6;$

$\mathfrak{n}_7^{110}:\ [X_1,X_i]=X_{i+1},\ i=2,3,5,\quad [X_2,X_3]=X_6,\ [X_5,X_7]=X_4;$

$\mathfrak{n}_7^{111}:\ [X_1,X_i]=X_{i+1},\ i=2,3,5,\quad [X_2,X_7]=X_6,\ [X_2,X_5]=X_4;$

$\mathfrak{n}_7^{112}:\ [X_1,X_i]=X_{i+1},\ i=2,3,5,\quad [X_2,X_3]=X_6,\ [X_2,X_7]=X_4;$

$\mathfrak{n}_7^{113}:\ [X_1,X_i]=X_{i+1},\ i=2,3,5,\quad [X_5,X_7]=X_6,\ [X_5,X_6]=X_4;$

$\mathfrak{n}_7^{114}:\ [X_1,X_i]=X_{i+1},\ i=2,3,5,\quad [X_2,X_7]=X_4,\ [X_5,X_6]=X_4,\ [X_5,X_7]=X_6;$

$\mathfrak{n}_7^{115}:\ [X_1,X_i]=X_{i+1},\ i=2,3,5,\quad [X_2,X_5]=X_4,\ [X_5,X_7]=X_3,\ [X_6,X_7]=X_4;$

$\mathfrak{n}_7^{116}:\begin{cases}[X_1,X_i]=X_{i+1},\ i=2,3,5,\quad [X_3,X_5]=-X_4,\ [X_2,X_6]=X_4,\\ [X_5,X_7]=-X_4;\end{cases}$

$$\mathfrak{n}_7^{117}(\alpha):\left\{\begin{array}{l}[X_1,X_i]=X_{i+1},\ i=2,3,5,\quad [X_2,X_5]=X_7,\ [X_2,X_7]=X_4,\\ [X_5,X_6]=X_4,\ [X_5,X_7]=\alpha X_4;\end{array}\right.$$

$$\mathfrak{n}_7^{118}:\left\{\begin{array}{l}[X_1,X_i]=X_{i+1},\ i=2,3,5,\quad [X_2,X_5]=X_7,\ [X_2,X_6]=X_4,\\ [X_3,X_5]=-X_4,\ [X_5,X_7]=-\frac{1}{4}X_4;\end{array}\right.$$

$$\mathfrak{n}_7^{119}:\ [X_1,X_2]=X_3,\ [X_1,X_3]=X_4,\ [X_2,X_5]=X_4,\ [X_6,X_7]=X_4.$$

Remarks.

(1) We can generalize this process to define $(n-1)$-associative (binary) algebras: we consider the relations defining the Σ_n-module $\mathcal{A}ss(n)$ of the quadratic operad $\mathcal{A}ss$ and define, as above, an algebra with a multiplication which is a bilinear map μ, satisfying a relation where μ occurs $n-1$ times in each term. This algebra will be called $(n-1)$-associative (binary) algebra. If the Lie bracket of a algebra $\mathfrak{g}$ is also $(n-1)$-associative, we prove a similar way than for the cubic associative case that $\mathfrak{g}$ is a nilpotent Lie algebra of nilindex $n-1$.

(2) There exits another notion of associativity for n-ary algebras (an n-ary algebra is a vector space with a multiplication which is an n-linear map), the total associativity. For example, a totally associative 3-ary algebra has a ternary multiplication, denoted xyz, satisfying the relation:

$$(xyz)tu = x(yzt)u = xy(ztu)$$

for any x,y,z,t,u. The corresponding operad is studied in [15], [16] and [11]. Let $\mathfrak{g}$ be a Lie algebra. We have the notion of Lie triple product given by $[[x,y],z]$. If we consider the vector space $\mathfrak{g}$ provided with the 3-ary product given by the Lie triple product, then $\mathfrak{g}$ is a 3-Lie algebra ([8] or [1]). Let us suppose now that the Lie triple bracket of $\mathfrak{g}$ is a totally associative product. This implies

$$[[[[X,Y],Z],T],U]=[[X,[[Y,Z],T]],U]=[[X,Y],[[Z,T],U]].$$

But

$$\begin{array}{rcl}[[X,Y],[[Z,T],U]] & = & -[[[Z,T],U],[X,Y]]\\ & = & [X,[Y,[[Z,T],U]]]+[Y,[[[Z,T],U],X]]\\ & = & [[[[Z,T],U],Y],X]-[[[[Z,T],U],X],Y]\\ & = & [[Z,[[T,U],Y]],X]-[[Z,[[T,U],X]],Y].\end{array}$$

We deduce

$$\begin{array}{rcl}[[X,[[Y,Z],T]],U] & = & [[Z,[[T,U],Y]],X]-[[Z,[[T,U],X]],Y]\\ & = & 2^5[[X,[[Y,Z],T]],U]-2^5[[X,[[Y,Z],U]],T]\end{array}$$

Then

$$(2^5-1)[[X,[[Y,Z],T]],U]=2^5[[X,[[Y,Z],U]],T].$$

This implies

$$[[X,[[Y,Z],T]],U]=0=[[[[X,Y],Z],T],U]=[[X,Y],[[Z,T],U]].$$

The Lie algebra is 4-step nilpotent.

3. Cubic operads

Let E be a $\mathbb{K}[\Sigma_2]$-module and $\Gamma(E)$ the free operad generated by E. Consider a $\mathbb{K}[\Sigma_4]$-submodule R of $\Gamma(E)(4)$. Let $\mathcal{R}$ the ideal of $\Gamma(E)$ generated by R. We have

$$\mathcal{R} = \{\mathcal{R}(n), n \in \mathbb{N}^*\}$$

with $\mathcal{R}(1) = \{0\},\ \mathcal{R}(2) = \{0\},\ \mathcal{R}(3) = \{0\},\ \mathcal{R}(4) = R.$

Definition 6. *We call cubic operad generated by E and defined by the relations $R \subset \Gamma(E)(4)$, the operad $\mathcal{P}(E,R)$ given by*

$$\mathcal{P}(E,R)(n) = \frac{\Gamma(E)(n)}{\mathcal{R}(n)}.$$

The operad $\mathcal{A}ss\mathcal{C}ubic$ is the cubic operad generated by $E = \mathbb{K}[\Sigma_2]$ and the $\mathbb{K}[\Sigma_4]$-submodule of relations R generated by the vectors

$$\left\{ \begin{array}{l} ((x_1x_2)x_3)x_4 - (x_1(x_2x_3))x_4, (x_1(x_2x_3))x_4 - x_1((x_2x_3)x_4), x_1((x_2x_3)x_4) - x_1(x_2(x_3x_4)), \\ x_1(x_2(x_3x_4)) - (x_1x_2)(x_3x_4), (x_1x_2)(x_3x_4) - ((x_1x_2)x_3)x_4. \end{array} \right.$$

Thus we have $\mathcal{A}ss\mathcal{C}ubic(2) = \mathbb{K}[\Sigma_2]$, $\mathcal{A}ss\mathcal{C}ubic(3) = \mathbb{K}[\Sigma_3]$, and

$$\mathcal{A}ss\mathcal{C}ubic(4) = \frac{\Gamma(E)(4)}{\mathcal{R}(4)}$$

is the 24-dimensional $\mathbb{K}$-vector space generated by $\{((x_\sigma(1)x_\sigma(2))x_\sigma(3))x_\sigma(4),\ \sigma \in \Sigma_4\}$.

The operad $3\mathcal{N}ilp$ is the cubic operad defined by $3\mathcal{N}ilp(2) = sgn_2$, $3\mathcal{N}ilp(3) = sgn_3 \oplus V_2/sgn_3$ and $3\mathcal{N}ilp(4) = \{0\}$.

Proposition 7. *The cubic operad $3\mathcal{N}ilp$ is not Koszul.*

Proof. In fact the Koszul operads may only quadratic relations.

Remark: The Jordan operad. There is a cubic operad which is really interesting, it is the operad $\mathcal{J}ord$ corresponding to Jordan algebras. Recall that a $\mathbb{K}$-Jordan algebra is a commutative algebra satisfying the following identity

$$x(yx^2) = (xy)x^2.$$

Since $\mathbb{K}$ is of zero characteristic, linearizing this identity, we obtain

$$((x_2x_3)x_4)x_1 + ((x_3x_1)x_4)x_2 + ((x_1x_2)x_4)x_3 - (x_2x_3)(x_4x_1) - (x_3x_1)(x_4x_2) - (x_1x_2)(x_4x_3) = 0.$$

This relation is cubic. It is invariant by the permutations $\tau_{12}, \tau_{13}, \tau_{23}, c, c^2$ where c is the cycle (123). Thus the $\mathbb{K}[\Sigma_4]$-module $\mathcal{R}(4)$ generated by the vector

$$((x_2x_3)x_4)x_1 + ((x_3x_1)x_4)x_2 + ((x_1x_2)x_4)x_3 - (x_2x_3)(x_4x_1) - (x_3x_1)(x_4x_2) - (x_1x_2)(x_4x_3)$$

is a vector space of dimension 4. Let us consider the cubic operad $\mathcal{J}ord$ define by

$$\mathcal{J}ord(2) = 1\!\!1,\ \mathcal{J}ord(3) = 1\!\!1 \oplus V,\ \mathcal{J}ord(4) = \frac{\Gamma(1\!\!1)(4)}{\mathcal{R}(4)}$$

where $1\!\!1$ is the identity representation of Σ_2. Since $\dim \Gamma(1\!\!1)(4) = 15$, thus $\dfrac{\Gamma(1\!\!1)(4)}{\mathcal{R}(4)}$ is of dimension 11.

If the Lie bracket of a Lie algebra $\mathfrak{g}$ is also a Jordan product, then $\mathfrak{g}$ is abelian. But maybe, it would be interesting to look Lie bracket satisfying the Jordan identity without the commutativity identity.

4. Generalized associative Lie multiplication

Since we always find 2-step nilpotent algebras when we consider Lie bracket satisfying associative or some non associative properties, the main idea is to generalize the notion of associativity. Different ways can be considered. A multiplication μ is associative if it satisfies

$$\mu \circ (\mu \otimes Id) - \mu \circ (Id \otimes \mu) = 0.$$

Let ϕ be a bilinear mapping. We can consider that the multiplication μ satisfies

$$\mu \circ (\phi \otimes Id) - \mu \circ (Id \otimes \phi) = 0.$$

This relation is based on the Hochschild cohomology of associative algebras. We find such a relation when we consider rigid associative multiplications with non trivial Hochschild cohomology. Such algebras exist, see the works of Mary Schaps [18] (In case of Lie algebras we have similar situations see [3, 9]). A particular case is given by considering for ϕ a bilinear map of type $\mu(f(x), y)$ where f is an endomorphism. In this paper we consider the following generalization of the notion of associativity, close to the notion of Hom-algebras introduced by the neighbor of our offices [10].

Definition 8. *Let A be a $\mathbb{K}$-algebra with multiplication μ and let f be a linear endomorphism of the vector space A. We say that A is a f-associative algebra if*

$$\mu \circ (\mu \otimes f) - \mu \circ (f \otimes \mu) = 0.$$

If we write $\mu(x \otimes y) = xy$, this previous identity corresponds to

$$(xy)f(z) = f(x)(yz)$$

for every $x, y, z \in A$. Let us consider a Lie algebra $\mathfrak{g}$ such that the bracket is also f-associative. If $f = Id$, we have seen that $\mathfrak{g}$ is 2-step nilpotent. In the general case, the Lie bracket satisfies the relation

$$[[X, Y], f(Z)] = [f(X), [Y, Z]]$$

for any $X, Y, Z \in \mathfrak{g}$. We deduce

$$\begin{array}{rcl} [f(X), [Y, Z]] & = & -[f(Z), [X, Y]] = [f(Y), [Z, X]] \\ & = & -[f(X), [Y, Z]], \end{array}$$

which is equivalent to

$$[f(X), [Y, Z]] = 0, \ \forall X, Y, Z \in \mathfrak{g}.$$

Proposition 9. *Let f be a linear endomorphism of the Lie algebra $\mathfrak{g}$. The Lie bracket of $\mathfrak{g}$ is f-associative if and only if f is with values in the center of the derived subalgebra.*

Corollary 10. (1) *A semi-simple Lie algebra is never f-associative.*
(2) *Let $\mathfrak{g}$ be a 2-step nilpotent Lie algebra. Then the bracket is f-associative for any linear endomorphism f of $\mathfrak{g}$.*

Consider
$$A_{\mathfrak{g}} = \{f \in End(\mathfrak{g}),\ f(\mathfrak{g}) \subset \mathcal{D}(\mathfrak{g})\}$$
where $\mathcal{D}(\mathfrak{g}) = \mathcal{C}^1(\mathfrak{g})$ is the derived subalgebra of $\mathfrak{g}$. It is clear that if f_1 and f_2 belong to $A_{\mathfrak{g}}$, then $f_1 \circ f_2$ also belongs to $A_{\mathfrak{g}}$. Thus $A_{\mathfrak{g}}$ is an associative algebra and also a Lie algebra.

Examples

(1) Let us consider the 2-dimensional solvable Lie algebra whose bracket satisfies
$$[e_1, e_2] = e_2.$$
Every endomorphism belonging to $A_{\mathfrak{g}}$ is written in a matricial form as:
$$\begin{pmatrix} 0 & 0 \\ b & d \end{pmatrix}.$$
The Lie algebra $A_{\mathfrak{g}}$ is isomorphic to $\mathfrak{g}$.

(2) Let us consider the n-dimensional filiform nilpotent Lie algebra whose bracket satisfies
$$[e_1, e_i] = e_{i+1}, \quad i = 2, \cdots, n-1.$$
If $f \in A_{\mathfrak{g}}$, then it is written:
$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ a_{21} & a_{22} & \cdots & 0 & a_{2,n} \\ a_{31} & a_{32} & \cdots & a_{3n-1} & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn-1} & a_{nn} \end{pmatrix}$$
It is an $(n^2 - n)$-dimensional solvable Lie algebra.

Remark. The notion of Hom-associative algebra introduced in [10] is a little be different from the notion of f-associativity. Recall the notion of Hom-associativity that we will denote by Hom-f-associativity since it depends on an endomorphism f:

Let A be a $\mathbb{K}$-algebra whose multiplication is denoted by xy and let f be a linear endomorphism of the vector space A. We say that A is a Hom-f-associative algebra if
$$(f(x)y)z = x(yf(z)),$$
for any $x, y, z \in A$.

If we denote by μ the multiplication in A, this relation writes as:
$$\mu \circ (\mu \otimes Id) \circ (f \otimes Id_2) = \mu \circ (Id \otimes \mu) \circ (Id_2 \otimes f).$$
Thus we can show that if A is a Hom-f-associative algebra and B a Hom-g-associative algebra, thus $A \otimes B$ with the natural tensor product is a Hom-$(f \otimes g)$-associative algebra. Let us consider a Lie algebra $\mathfrak{g}$. If the Lie bracket is Hom-f-associative, thus
$$[[f(X), Y], Z] = [X, [[Y, f(Z)]]$$
or
$$[[f(X), Y], Z] = [[f(Z), Y], X].$$
We will denote by $B_{\mathfrak{g}}$ the vector space
$$B_{\mathfrak{g}} = \{f \in End(\mathfrak{g}), [[f(X), Y], Z] = [[f(Z), Y], X], X, Y, Z \in \mathfrak{g}.\}$$

Examples.

(1) Let us consider the 2-dimensional solvable Lie algebra whose bracket satisfies
$$[e_1, e_2] = e_2.$$
Every endomorphism of $\mathfrak{g}$ such that the bracket is an Hom-f-associative multiplication is written in a matricial form as:
$$\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$$

(2) Let us consider the n-dimensional filiform nilpotent Lie algebra whose bracket satisfies
$$[e_1, e_i] = e_{i+1}, \quad i = 2, \cdots, n-1.$$
Every endomorphism of $\mathfrak{g}$ such that the bracket is an Hom-f-associative multiplication is written in a matricial form as:
$$\begin{pmatrix} a_{11} & 0 & \cdots & 0 & 0 \\ a_{21} & 0 & \cdots & 0 & a_{2,n} \\ a_{31} & a_{32} & \cdots & a_{3n-1} & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn-1} & a_{nn} \end{pmatrix}$$
In fact, let f whose matrix in the basis $\{e_i\}$ is as above. If $i \neq 1$, $[f(e_i), e_j] = 0$. Thus only the identities $[[f(e_1), e_j], e_k] = [[f(e_k), e_j], e_1]$ have to be verified. This is the case.

(3) Let $\mathfrak{g}$ be the simple Lie algebra $sl(2, \mathbb{C})$. There exists no endomorphism f such that the bracket of $sl(2, \mathbb{C})$ is Hom-f-associative.

Proposition 11. *Assume that $\mathfrak{g}$ is 2-step solvable, and let f be an endomorphism of the derived subalgebra $D(\mathfrak{g})$. Thus $\mathfrak{g}$ is a Lie algebra Hom-f-associative.*

For example, if $\mathfrak{g}$ is the model filiform Lie algebra described in the previous example 2, it is $(n-1)$-step nilpotent but 2-step solvable. Thus any endomorphism
$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & a_{3n-1} & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn-1} & a_{nn} \end{pmatrix}$$
provided $\mathfrak{g}$ with a Hom-f-associative structure. The set of such matrices is a Lie algebra included in $B_{\mathfrak{g}}$.

Assume now that $\mathfrak{g}$ is a complex simple Lie algebra. We have seen that $sl(2, \mathbb{C})$ cannot be endowed with a Hom-f-associative structure. If the rank of $\mathfrak{g}$ is greater than or equal to 2, we can consider two independent vectors H_1 and H_2 in the Cartan subalgebra of $\mathfrak{g}$. For any $X \in \mathfrak{g}$, we have
$$[[f(H_1), X], H_2] = [[f(H_2), X], H_1].$$
Thus $[f(H_1), X] = [f(H_2), X] = 0$ for any $X \in \mathfrak{g}$. As $\mathfrak{g}$ is simple, we deduce $f(H_1) = f(H_2) = 0$ and f is trivial on the Cartan subalgebra. Thus
$$[[f(X), Y], H] = [[f(H), Y], X] = 0$$

for any $X, Y \in \mathfrak{g}$. So $[f(X), Y] = 0$ for any $Y \in \mathfrak{g}$ and $f(X) = 0$. The endomorphism f is trivial on $\mathfrak{g}$.

Proposition 12. *Let $\mathfrak{g}$ be a simple complex Lie algebra. Any linear endomorphism f which provides the Lie bracket of $\mathfrak{g}$ with a Hom-f-associative structure is trivial.*

Thus, in this case $B_{\mathfrak{g}} = 0$. In the general case, $B_{\mathfrak{g}}$ is a vector space and the composition map is not a multiplication on $B_{\mathfrak{g}}$. It is also not a Lie algebra. We will examine some properties of $B_{\mathfrak{g}}$. If $f \in A_{\mathfrak{g}}$, the bracket satisfies the relation

$$[[f(X), Y], Z] = [X, [[Y, f(Z)]].$$

As the bracket is skew-symmetric, we obtain

$$[Z, [f(X), Y]] = [X, [f(Z), Y]],$$

that is,

$$ad(Z) \circ ad(f(X)) = ad(X) \circ ad(f(Z))$$

for any $X, Z \in \mathfrak{g}$. Let $K_{\mathfrak{g}}$ be the Killing-Cartan form of $\mathfrak{g}$, then

$$K_{\mathfrak{g}}(f(X), Z) = K_{\mathfrak{g}}(X, f(Z))$$

and we have that f is a symmetric map for $K_{\mathfrak{g}}$.

Proposition 13. *Any $f \in B_{\mathfrak{g}}$ satisfies*

$$[[X, Y], f(Z)] + [[Y, Z], f(X)] + [[Z, X], f(Y)] = 0$$

for any $X, Y, Z \in \mathfrak{g}$.

In fact

$$\begin{array}{rcl}
[[f(X), Y], Z] & = & [X, [[Y, f(Z)]] \\
& = & -[Y, [[f(Z), X]] - [f(Z), [[X, Y]] \\
& = & [[X, Y], f(Z)] + [[f(Y), X], Z] \\
& = & [[X, Y], f(Z)] - [[X, Z], f(Y)] - [[Z, f(Y)], X] \\
& = & [[X, Y], f(Z)] + [[Z, X], f(Y)] + [[f(X), Z], Y] \\
& = & [[X, Y], f(Z)] + [[Z, X], f(Y)] - [[Z, Y], f(X)] - [[Y, f(X)], Z] \\
& = & [[X, Y], f(Z)] + [[Z, X], f(Y)] + [[Y, Z], f(X)] + [[f(X), Y], Z].
\end{array}$$

Thus $[[X, Y], f(Z)] + [[Z, X], f(Y)] + [[Y, Z], f(X)] = 0$. We have also relations in greater degree.

Proposition 14. *For any $f_1, f_2, f_3 \in B_{\mathfrak{g}}$ we have*

$$\begin{array}{l}
[[f_1 \circ f_2 \circ f_3(X_1), X_2], X_3] - [[f_3 \circ f_2 \circ f_1(X_3), X_2], X_2] + [[f_2 \circ f_3 \circ f_1(X_3), X_1], X_2] \\
-[[f_1 \circ f_2 \circ f_3(X_1), X_2], X_3] + [[f_3 \circ f_1 \circ f_2(X_2), X_3], X_1] - [[f_2 \circ f_1 \circ f_3(X_1), X_3], X_2] \\
= 0
\end{array}$$

for any $X_1, X_2, X_3 \in \mathfrak{g}$.

In fact

$$\begin{array}{rcl}
[[f_1 \circ f_2 \circ f_3(X_1), X_2], X_3] & = & [[f_1(X_3), X_2], f_2 \circ f_3(X_1)] \\
 & = & -[[X_2, \circ f_2 \circ f_3(X_1)], f_1(X_3)] - [[\circ f_2 \circ f_3(X_1),], X_2] \\
 & = & [[f_2 \circ f_3(X_1), X_2], f_1(X_3)] - [[f_2(X_2), f_1(X_3)], f_3(X_1)] \\
 & = & -[[f_2 \circ f_1(X_3), X_2], f_3(X_1)] + [[f_1(X_3), f_2(X_2)], f_3(X_1)] \\
 & = & -[[X_2, f_3(X_1)], f_2 \circ f_1(X_3)] - [[f_3(X_1), f_2 \circ f_1(X_3), X_2] \\
 & & +[[f_1(X_3), f_2(X_2)], f_3(X_1)] \\
 & = & [[f_3 \circ f_2 \circ f_1(X_3)], X_1] + [[f_2(X_2), f_3(X_1)], f_1(X_3)] \\
 & & +[[f_1(X_3), f_2(X_2)], f_3(X_1)]
\end{array}$$

We deduce

$$[[f_1 \circ f_2 \circ f_3(X_1), X_2], X_3] - [[f_3 \circ f_2 \circ f_1(X_3), X_2], X_1] + [[f_3(X_1), f_1(X_3)], f_2(X_2)] = 0.$$

We have also

$$[[f_2 \circ f_3 \circ f_1(X_3), X_1], X_2] - [[f_1 \circ f_3 \circ f_2(X_2), X_1], X_3] + [[f_1(X_3), f_2(X_2)], f_3(X_1)] = 0,$$

and

$$[[f_3 \circ f_1 \circ f_2(X_2), X_3], X_1] - [[f_2 \circ f_1 \circ f_3(X_1), X_3], X_2] + [[f_2(X_2), f_3(X_1)], f_1(X_3)] = 0.$$

If we add these three relations, we obtain the result.

Acknowledgement. We thank Vladimir Dotsenko to point out a mistake in the first version.

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